LOWER SEMI-CONTINUITY OF INTEGRALS WITH
G-QUASICONVEX POTENTIAL

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ABSTRACT. This paper introduces the proper notion of variational quasiconvexity associated to a group of diffeomorphisms. We prove a lower semicontinuity theorem connected to this notion. In the second part of the paper we apply this result to a class of functions, introduced in [5]. Such functions are $GL(n,R)^+$ quasiconvex, hence they induce lower semicontinuous integrals.

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1. INTRODUCTION

Lower semi-continuity of variational integrals

$$u \ \mapsto \ I(u) = \int_\Omega w(Du(x)) \ \mathrm{d}x$$

defined over Sobolev spaces is connected to the convexity of the potential w. In the scalar case, that is for functions u with domain or range in R, the functional I is weakly $W^{1,p}$ lower semi-continuous (weakly $* \ W^{1,\infty}$) if and only if w is convex, provided it is continuous and satisfies some growth conditions. The notion which replaces convexity in the vector case is quasi-convexity (introduced by Morrey [14]).

We shall concentrate on the case $u : \Omega \subset R^n \to R^n$ which is interesting for continuum media mechanics. Standard notation will be used, like:

$gl(n,R)$	the linear space (Lie algebra) of $n \times n$ real matrices
$GL(n,R)$	the group of invertible $n \times n$ real matrices
$GL(n,R)^+$	the group of matrices with positive determinant
$sl(n,R)$	the algebra of traceless $n \times n$ real matrices
$SL(n,R)$	the group of real matrices with determinant one
$CO(n)$	the group of conformal matrices
id	the identity map
1	the identity matrix
$\circ$	function composition

In this frame Morrey's quasiconvexity has the following definition.

Definition 1.1. *Let $\Omega \subset R^n$ be an open bounded set such that $\mid \partial\Omega \mid = 0$ and $w : gl(n,R) \to R$ be a measurable function. The map w is quasiconvex if for any $H \in gl(n,R)$ and any Lipschitz $\eta : \Omega \to R^n$, such that $\eta(x) = 0$ on $\partial\Omega$, we have*

$$(1) \qquad \int_\Omega w(H) \ \leq \ \int_\Omega w(H + D\eta(x))$$

Translation and rescaling arguments show that the choice of Ω is irrelevant in the above definition.

Any quasiconvex function w is rank one convex. There are several ways to define rank one convexity but this is due to the regularity assumptions upon w. The most

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natural, physically meaningful and historically justified, is to suppose that w is C^2 and link rank one convexity with the ellipticity (cf. Hadamard [10]) of the Euler-Lagrange equation associated to w. There are well-known ways to show that one can get rid of any regularity assumption upon w, replacing it by some growth conditions. Rank one convexity becomes then just what the denomination means, that is convexity along any rank one direction.

Proposition 1.1. *Suppose that* $w : gl(n, R) \to R$ *is* C^2 *and quasiconvex. Then for any pair* $a, b \in R^n$ *the ellipticity inequality*

$$(2) \qquad \frac{\partial^2 w}{\partial H_{ij} \partial H_{kl}}(H) a_i b_j a_k b_l \ \geq \ 0$$

holds true.

Proof. Take any $\eta \in C^2(\Omega, R^n)$ such that $\eta(x) = 0$ on $\partial\Omega$ and $H \in gl(n, R)$. If w is quasiconvex then the function

$$t \mapsto \ f(t) \ = \ \int_\Omega w(H + tD\eta(x))$$

has a minimum in $t = 0$. Therefore $f'(0) = 0$ and $f''(0) \geq 0$. Straightforward computation shows that $f'(0) = 0$ anyway and $f''(0) \geq 0$ reads:

$$\frac{\partial^2 w}{\partial H_{ij} \partial H_{kl}}(H) \int_\Omega \eta_{i,j}(x) \eta_{k,l}(x) \ \geq \ 0$$

With the notation

$$\Delta(\eta) \ = \ \int_\Omega D\eta(x) \otimes D\eta(x)$$

remark that $\Delta(x) \in V = gl(n, R) \otimes gl(n, R)$, because V is a vectorspace and $D\eta(x) \otimes D\eta(x) \in V$ for any $x \in \Omega$. It follows that there is $P \in gl(n, R)$ such that:

$$\Delta(\eta)_{ijkl} = P_{ij} P_{kl}$$

Integration by parts shows that $\Delta(\eta)$ has more symmetry, namely:

$$\Delta(\eta)_{ijkl} = \Delta(\eta)_{ilkj}$$

which turns to be equivalent to $rank \ P \ \leq 1$. Therefore there are $a, b \in R^n$ such that $P = a \otimes b$.

All it has been left to prove is that for any $a, b \in R^n$ there is a $\lambda \neq 0$ and a vector field $\eta \in C^2(\Omega, R^n)$ such that $\eta(x) = 0$ on $\partial\Omega$ and $\Delta(\eta) = \lambda a \otimes b$. For this suppose that Ω is the unit ball in R^n, take $u : [0, \infty] \to R$ a C^∞ map, such that $u(1) = 0$ and define:

$$\eta(x) = u(\mid x \mid^2) \sin(b \cdot x) a$$

It is a matter of computation to see that η is well chosen to prove the thesis. $\qquad\square$

In elasticity the elastic potential function w is not defined on the Lie algebra $gl(n, R)$ but on the Lie group $GL(n, R)$ or a subgroup of it. It would be therefore interesting to find the connections between lower semicontinuity of the functional and the (well chosen notion of) quasiconvexity in this non-linear context. This is a problem which floats in the air for a long time. Let us recall two different definitions of quasiconvexity which are relevant.

Definition 1.2. *Let* $w : GL(n, R)^+ \to R$. *Then:*

  (a) *(Ball [2]) w is quasiconvex if for any $F \in GL(n, R)^+$ and any $\eta \in C_c^\infty(\Omega, R^n)$ such that $F + D\eta(x) \in GL(n, R)^+$ for almost any $x \in \Omega$ we have*

$$\int_\Omega w(F + D\eta(x)) \ \geq \ \mid \Omega \mid w(F)$$

(b) *(Giaquinta, Modica & Soucek [9], page 174, definition 3) w is Diff-quasiconvex if for any diffeomorphism $\phi : \Omega \to \phi(\Omega)$ such that $\phi(x) = Fx$ on $\partial\Omega$, for some $F \in GL(n, R)^+$ we have:*

$$\int_\Omega w(D\phi(x)) \; \geq \; \int_\Omega w(F)$$

These two definitions are equivalent.

It turns out that very little is known about the lower semicontinuity properties of integrals given by Diff-quasiconvex potentials. It is straightforward that Diff-quasiconvexity is a necessary condition for weakly $* W^{1,\infty}$ (or uniform convergence of Lipschitz mappings) (see [9] proposition 2, same page). All that is known reduces to the properties of polyconvex maps. A polyconvex map $w : GL(n, R)^+ \to R$ is described by a convex function $g : D \subset R^M \to R$ (the domain of definition D is convex as well) and M rank one affine functions $\nu_1, ..., \nu_M : GL(n, R)^+ \to R$ such that for any $F \in GL(n, R)^+$

$$w(F) = g(\nu_1(F), ..., \nu_M(F))$$

The rank one affine functions are known(cf. Edelen [7], Ericksen [8], Ball, Curie, Olver [4]): ν is rank one affine if and only if $\nu(F)$ can be expressed as a linear combination of subdeterminants of F (uniformly with respect to F). Any rank one convex function is also called a null Lagrangian, because it generates a trivial Euler-Lagrange equation.

Polyconvex function give lower semicontinuous functionals, as a consequence of Jensen's inequality and continuity of (integrals of) null lagrangians. This is a very interesting path to follow (cf. Ball [3]) and it leads to many applications. But it leaves unsolved the problem: are the integrals given by Diff-quasiconvex potentials lower semicontinuous?

In the case of incompressible elasticity one has to work with the group of matrices with determinant one, i.e. $SL(n, R)$. The "linear" way of thinking has been compensated by wonders of analytical ingenuity. One purpose of this paper is to show how a slight modification of thinking, from linear to nonlinear, may give interesting results in the case $w : G \to R$ where G is a Lie subgroup of $GL(n, R)$. Note that when n is even a group which deserves attention is $Sp(n, R)$, the group of symplectic matrices.

From now on linear transformations of R^n and their matrices are identified. G is a Lie subgroup of $GL(n, R)$.

Definition 1.3. *For any $\Omega \subset R^n$ open, bounded, with smooth boundary, we introduce the set $[G]^\infty(\Omega)$ of all bi-Lipschitz mappings u from Ω to R^n such that for almost any $x \in \Omega$ we have $Du(x) \in G$.*

The set $Q \subset R^n$ is the unit cube $(0, 1)^n$.
The departure point of the paper is the following natural definition.

Definition 1.4. *The continuous function $w : G \to R$ is G-quasiconvex if for any $F \in G$ and $u \in [G]^\infty(Q)$ we have:*

(3)
$$\int_Q w(F) \; dx \; \leq \; \int_Q w(FDu(x)) \; dx$$

We describe now the structure of the paper. After the formulation of the lower semicontinuity theorem 2.1, in section 3 is shown that quasiconvexity in the sense of definition 1.2 is the same as $GL(r, n)^+$ quasiconvexity. Theorem 2.1 is proved in section 4; in the next section is described the rank one convexity (or ellipticity)

notion associated to G quasiconvexity. The cases $GL(n, R)$ and $SL(n, R)$ are examined in detail. It turns out that classification of all universal conservation laws in incompressible elasticity is based on some unproved assumptions. In section 6 is described a class of $GL(n, R)^+$ quasiconvex functions introduced in Buliga [5]. Theorem 2.1 is used to prove that any such function induces a lower semicontinuous integral.

2. G-QUASICONVEXITY AND THE LOWER SEMICONTINUITY RESULT

We denote by $[G]_c^\infty$ the class of all Lipschitz mapping from R^n to R^n such that $u - id$ has compact support and for almost any $x \in R^n$ we have $Du(x) \in G$. The main result of the paper is:

Theorem 2.1. *Let G be a Lie subgroup of $GL(n, R)$, Ω an open, bounded set with $\mid \partial\Omega \mid = 0$ and $w : G \to R$ locally Lipschitz.*

  a) *Suppose that for any sequence $u_h \in [G]_c^\infty$ weakly $*$ $W^{1,\infty}$ convergent to id we have:*

$$(4) \qquad \int_\Omega w(F) \ dx \ \leq \ \liminf_{h\to\infty} \int_\Omega w(FDu_h(x)) \ dx$$

  Then for any bi-Lipschitz $u \in [G]_c^\infty$ and for any sequence u_h weakly $*$ $W^{1,\infty}$ convergent to u we have:

$$(5) \qquad \int_\Omega w(Du(x)) \ dx \ \leq \ \liminf_{h\to\infty} \int_\Omega w(Du_h(x)) \ dx$$

  Moreover, if (5) holds for any bi-Lipschitz $u \in [G]_c^\infty$ and for any sequence u_h weakly $*$ $W^{1,\infty}$ convergent to u then w is G-quasiconvex.

  b) *Suppose that G contains the group $CO(R^n)$ of conformal matrices. Then (5) holds for any bi-Lipschitz $u \in [G]_c^\infty$ and for any sequence u_h weakly $*$ $W^{1,\infty}$ convergent to u if and only if w is G-quasiconvex.*

The fact that weakly $*$ lower semicontinuity implies G quasiconvexity (end of point (a)) is easy to prove by rescaling arguments (cf. proposition 2, Giaquinta, Modica and Soucek *op. cit.*).

The method of proving the point (a) of the theorem is well known (see Meyers [13]). Even if there is nothing new there from the pure analytical viewpoint, I think that the proof deserves attention.

3. G-QUASICONVEXITY

This section contains preliminary properties of G-quasiconvex continuous functions.

Proposition 3.1.   a) *In the definition of G-quasiconvexity the cube Q can be replaced by any open bounded set Ω such that $\mid \partial\Omega \mid = 0$.*

  b) *The function w is G-quasiconvex if and only if for any $F \in G$ and $u \in [G]_c^\infty(Q)$ we have:*

$$(6) \qquad \int_Q w(F) \ dx \ \leq \ \int_Q w(Du(x)F) \ dx$$

  The converse is true.

  c) *For any $U \in GL_n$ such that $UGU^{-1} \subset G$ and for any $W : G \to R$ G-quasiconvex, the mapping $W_U : G \to R$, $W_U(F) = W(UFU^{-1})$ is G-quasiconvex.*

Remark 3.1. The point b) shows that the non-commutativity of the multiplication operation does not affect the definition of G-quasiconvexity. The point c) is a simple consequence of the fact that G is a group.

Proof. The point a) has a straightforward proof by translation and rescaling arguments.

For b) let us consider $F \in G$ and an arbitrary open bounded $\Omega \subset R^n$ with smooth boundary. The application which maps $\phi \in [G]_c^\infty(\Omega)$ to $F^{-1}\phi F \in [G]_c^\infty(F^{-1}(\Omega))$ is well defined and bijective. By a), if the function w is G-quasiconvex then we have

$$\int_{F^{-1}(\Omega)} w(FD(F^{-1}\phi F)(x)) \, \mathrm{d}x \geq \mid F^{-1}(\Omega) \mid w(F)$$

The change of variables $x = F^{-1}y$ resumes the proof of b).

With U like in the hypothesis of c), the application which maps $\phi \in [G]_c^\infty(\Omega)$ to $U\phi U^{-1} \in [G]_c^\infty(U^{-1}(\Omega))$ is well defined and bijective. The proof resumes as for the point b). $\qquad\square$

The following proposition shows that quasi-convexity in the sense of definition 1.2 is a particular case of G-quasiconvexity.

Proposition 3.2. *Let us consider $F \in GL(n, R)^+$. Then w is $GL(n, R)^+$-quasiconvex in F if and only if it is quasi-convex in F in the sense of Ball.*

Proof. Let $E \subset R^n$ be an open bounded set and $\phi \in [GL(n, R)^+]_c^\infty(E)$. The vector field $\eta = F(\phi - id)$ verifies the condition that almost everywhere $F + D\eta(x)$ is invertible. Therefore, if w is quasi-convex in F, we derive from the inequality:

$$\int_E w(FD\phi(y)) \, \mathrm{d}y \geq \mid E \mid W(F) \; .$$

We implicitly used the chain of equalities

$$F + D\eta(y) = F + FD\phi(y) - F = FD\phi(y) \; .$$

We have proved that quasi-convexity implies $GL(n, R)^+$-quasiconvexity.

In order to prove the inverse implication let us consider η such that almost everywhere $F + D\eta(x)$ is invertible. We have therefore $\phi = F^{-1}\psi \in [GL(n, R)^+](E)$ and $FD\phi = F + D\eta$. We use now the hypothesis that w is $GL(n, R)^+$-quasiconvex in F and we find that w is also quasi-convex. $\qquad\square$

4. Proof of Theorem 2.1

The proof is divided into three steps. In the first step we shall prove the following:

(Step 1.) *Let $w : GL(n, R) \to R$ be locally Lipschitz. Suppose that for any Lipschitz bounded sequence $u_h \in [GL(n, R)]_c^\infty$ uniformly convergent to id on $\overline{\Omega}$ and for any $F \in GL(n, R)$ we have:*

$$(7) \qquad \int_\Omega w(F) \, dx \leq \liminf_{h \to \infty} \int_\Omega w(FDu_h(x)) \, dx$$

Then for any bi-Lipschitz $u : R^n \to R^n$ and for any sequence $u_h \in [GL(n, R)]_c^\infty$ uniformly convergent to id on $\overline{\Omega}$ we have:

$$(8) \qquad \int_\Omega w(Du(x)) \, dx \leq \liminf_{h \to \infty} \int_\Omega w(D(u_h \circ u)(x)) \, dx$$

Remark 4.1. This is just the point a) of the main theorem for the whole group of linear invertible transformations.

Proof. For $\varepsilon > 0$ sufficiently small consider the set:

$$U^\varepsilon = \left\{ B = \overline{B}(x,r) \subset \Omega \ : \ \exists \ A \in GL(n,R) \ , \ \int_B | \ Du(x) - A \ | < \varepsilon \ | \ B \ | \right\}$$

From the Vitali covering theorem and from the fact that u is bi-Lipschitz we deduce that there is a sequence $Bj = \overline{B}(x_j, r_j) \in U^\varepsilon$ such that:

- $| \ \Omega \setminus \bigcup_j B_j \ | = 0$
- for any j u is approximatively differentiable in x_j and $Du(x_j) \in GL(n,R)$
- we have

$$\int_{B_j} | \ Du(x) - Du(x_j) \ | \ < \ \varepsilon \ | \ B_j \ |$$

Choose N such that

$$| \ \Omega \setminus \bigcup_{j=1}^{N} B_j \ | \ < \ \varepsilon$$

We have therefore:

$$\int_\Omega w(D(u_h \circ u)(x)) \ \geq \ \sum_{j=1}^{N} \int_{B_j} w(D(u_h \circ u)(x)) \ - \ C\varepsilon$$

$$\sum_{j=1}^{N} \int_{B_j} w(D(u_h \circ u)(x)) = J_1 + J_2 + J_3$$

where the quantities J_i are given below, with their estimates.

$$J_1 = \sum_{j=1}^{N} \int_{B_j} [w(Du_h(u(x))Du(x)) - w(Du_h(u(x))Du(x_j))]$$

$$| \ J_1 \ | \leq \sum_{j=1}^{N} \int_{B_j} | \ w(Du_h(u(x))Du(x)) - w(Du_h(u(x))Du(x_j)) \ | \ < \ C\varepsilon$$

$$J_2 = \sum_{j=1}^{N} \int_{B_j} [w(Du_h(u(x))Du(x_j)) - w(Du_h(\overline{u}_j(x))Du(x_j))]$$

where $\overline{u}_j(x) = u(x_j) + Du(x_j)(x - x_j)$. We have the estimate:

$$| \ J_2 \ | \leq C\varepsilon$$

Indeed, by changes of variables we can write:

$$I'_j \ = \ \int_{B_j} w(Du_h(u(x))Du(x_j)) \ = \ \int_{u(B_j)} w(Du_h(y)Du(x_j) \ | \det Du^{-1}(y) \ |$$

$$I_j\text{"} \ = \ \int_{B_j} w(Du_h(\overline{u}_j(x))Du(x_j)) \ = \ \int_{\overline{u}_j(B_j)} w(Du_h(y)Du(x_j) \ | \det(Du(x_j))^{-1} \ |$$

The difference $| \ I'_j - I_j\text{"} \ |$ is majorised like this

$$| \ I'_j - I_j\text{"} \ | \leq \int_{u(B_j) \cap \overline{u}_j(B_j)} C \ || \det Du^{-1}(y) \ | - | \det(Du(x_j))^{-1} \ || \ + C \ | \ u(B_j) \Delta \overline{u}_j(B_j) \ |$$

The function $| \det \cdot \ |$ is rank one convex and satisfies the growth condition $| \det F \ | \leq c(1 + | \ F \ |^n)$ for any $F \in GL(n,R)$. Therefore this function satisfies also the inequality:

$$|| \det F \ | - | \det P \ || \leq \ C \ | \ F - P \ | \left(1 + | \ F \ |^{n-1} + | \ P \ |^{n-1}\right)$$

Use now this inequality, the properties of the chosen Vitali covering and the uniform bound on Lipschitz norm of u, u_h, to get the claimed estimate.

$$J_3 = \sum_{j=1}^{N} \int_{B_j} w(Du_h(\overline{u}_j(x))Du(x_j))$$

By the change of variable $y = \overline{u}_j(x)$ and the hypothesis we have

$$\liminf_{h \to \infty} J_3 \geq \liminf_{h \to \infty} \sum_{j=1}^{N} \int_{B_j} w(Du(x_j))$$

Put all the estimates together and pass to the limit with $N \to \infty$ and then $\varepsilon \to 0$. $\qquad\blacksquare$

(**Step 2.**) If we replace in **Step 1.** *the group $GL(n,R)$ by a Lie subgroup G the conclusion is still true.*

Proof. Indeed, remark that in the proof of the previous step it is used only the fact that $GL(n,R)$ is a group of invertible maps. $\qquad\blacksquare$

Step 3. *The point b) of the Theorem 2.1 is true.*

Remark 4.2. In the classical setting of quasiconvexity, this step is proven by an argument involving Lipschitz extensions with controlled Lipschitz norm. In our case the corresponding Lipschitz extension assertion would be: *let $u \in [G]_c^\infty$ with Lipschitz norm $\|u - id\| = \varepsilon$. For $\delta > 0$ sufficiently big there exists $v \in [G](B(0, 1 + \delta))$ such that $v = u$ on $B(0,1)$ and $\|v - id\|$ controlled from above by ε.* This is not known to be true, even for $G = GL(n,R)$. That is why we shall use a different approach.

Proof. Because G is a group, it is sufficient to make the proof for $F = 1$.

Let $u_h \in [G]_c^\infty$ be a sequence weakly * convergent to id on Ω and $D \subset\subset \Omega$. For $\varepsilon > 0$ sufficiently small and $C > 1$ we have

$$D_{C\varepsilon} = \bigcup_{x \in D} B(x, C\varepsilon) \subset \Omega$$

It is not restrictive to suppose that

$$\lim_{h \to \infty} \int_{\Omega} w(Du_h) \, dx$$

exists and it is finite. For any $\varepsilon > 0$ there is N_ε such that for any $h > N_\varepsilon$ $u_h(D) \subset D_\varepsilon$.

Take a minimal Lipschitz extension

$$\overline{u}_h : D_{C\varepsilon} \setminus C \to R^n \ , \ \overline{u}_h(x) = \begin{cases} u_h(x) & , \ x \in \partial D \\ x & , \ x \in \partial D_{C\varepsilon} \end{cases}$$

The Lipschitz norm of this extension, denoted by k_h, is smaller than some constant independent on h.

Now, for any h define:

$$\psi_h = \frac{1}{2k_h} \overline{u}_{h|_{D_{C\varepsilon} \setminus D}}$$

According to Dacorogna-Marcellini Theorem 7.28, Chapter 7.4. [6], there is a solution σ_h of the problem

$$\begin{cases} D\sigma_h \in O(n) & \text{a. e. in } D_{C\varepsilon} \setminus D \\ \sigma_h = \psi_h & \text{on } \partial(D_\varepsilon \setminus D) \end{cases}$$

Let

$$v_h(x) = \begin{cases} u_h(x) & x \in D \\ k_h \sigma_h(x) & x \in \Omega \setminus D \end{cases}$$

Note that $Dv_h \in CO(n)$.

The following estimate is then true:

$$| \int_D w(Du_h) \ \mathrm{d}x \ - \ \int_\Omega w(Dv_h) \ \mathrm{d}x \ | \ = \ | \int_{D_{C\varepsilon} \setminus D} w(Dv_h) \ \mathrm{d}x \ | \ \leq$$

$$\leq \ \int_{D_{C\varepsilon} \setminus D} | \ w(Dv_h) \ | \ \mathrm{d}x \ \leq \ C \ | \ D_\varepsilon \setminus D \ |$$

w is G-quasiconvex, therefore:

$$\int_{D_\varepsilon} w(Dv_h) \ \mathrm{d}x \ \geq \ | \ D_\varepsilon \ | \ w(1)$$

We put all together and we get the inequality:

$$\lim_{h \to \infty} \int_D w(Du_h) \ \mathrm{d}x \ \geq \ | \ D_\varepsilon \ | \ w(1) - C \ | \ D_\varepsilon \setminus D \ |$$

The proof finishes after we pass ε to 0. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. RANK ONE CONVEXITY

The rank-one convexity notion associated to G quasi-convexity is described in the next proposition, for $w \in C^2(G, R)$. Before this, let us introduce a differential operator naturally connected to the group structure of G. Denote by $\mathcal{G}$ the Lie algebra of G. For any pair $(F, H) \in G \times \mathcal{G}$, the derivative of $w : G \to R$ in F with respect to H is

$$Dw(F)H \ = \ \frac{d}{dt}_{|t=0} w(F \exp(tH))$$

We shall also use the notation (for $F \in G$ and $H, P \in \mathcal{G}$):

$$D^2 w(F)(H, P) \ = \ D(Dw(\cdot)H)(F)P$$

Proposition 5.1. *A necessary condition for $w \in C^2(G, R)$ to be G quasi-convex is*

$$\int_\Omega D^2 w(F)(D\eta(x), D\eta(x)) \ = 0$$

for any $F \in G$ and $\eta \in C^2(\Omega, R^n)$, $D\eta(x) \in \mathcal{G}$ a.e. in Ω, supp $\eta \in \Omega$.

Proof. Given such an η, consider the solution of the o.d.e. problem:

$$\dot{\phi}_t \ = \eta \circ \phi_t \ , \ \ \phi_0 = \ id_{|\Omega}$$

This is an one-parameter group in the diffeomorphism class $[G]^\infty(\Omega)$. Define then:

$$f(t) \ = \ \int_\Omega w(FD\phi_t(x))$$

The G quasiconvexity of w implies that f has a minimum in $t = 0$. That means $f'(0) = 0$ and $f''(0) \geq 0$. The first condition is trivially satisfied and the second is, by straightforward computation, just the conclusion of the proposition. $\qquad\square$

We shall call G rank one convex a function which satisfies the conclusion of the proposition 5.1.

Consider the vector space

$$V(\mathcal{G}) = \{(H, H) \in \mathcal{G} \times \mathcal{G} \ : \ H \in \mathcal{G}\}$$

and the set

$$RO(\mathcal{G}) = \{(a, b) \in R^n \times R^n \ : \ a \otimes b \in \mathcal{G}\}$$

Proposition 5.2. *Suppose that $w : G \to R$ is a C^2 function. If for any $a, b \in RO(\mathcal{G})$*

$$(9) \qquad D^2 w(F)(a \otimes b, a \otimes b) \geq 0$$

then w is G rank one convex.

Proof. We shall use the notations from the proof of the preceding proposition. We see that

$$\int_\Omega (D\eta(x), D\eta(x)) \in V(\mathcal{G})$$

Therefore there is an $X \in \mathcal{G}$ such that

$$(X, X) = \int_\Omega (D\eta(x), D\eta(x))$$

Using integration by parts we find that for any indices $i, j, k, l \in 1, ..., n$ we have:

$$X_{ij} X_{kl} = X_{il} X_{kj}$$

which implies that X has rank one. Hence there are $a, b \in R^n$ such that $X = a \otimes b$. Use the definition of G rank one convexity to prove that (9) implies the G rank one convexity. $\qquad\square$

In the case $G = GL(n, R)$ we find that $GL(n, R)$ rank one convexity is equivalent to classical rank one convexity. To see this, take arbitrary $F \in GL(R^n)$, $a, b \in R^n$, $s > 0$ and $u \in C_c^\infty(\Omega, R)$. Define

$$\eta^s(x) = u(x) \sin [s(b \cdot x)]\ a$$

Because $GL(n, R)$ is an open set in the vectorspace of $n \times n$ real matrices, the $GL(n, R)$ rank one condition reads:

$$s^2 \ \frac{d^2 w}{dF_{ij} dF_{kl}}(F)(Fa)_i b_j (Fa)_k b_l \int_\Omega u^2\ + B\ \geq 0$$

with B independent on s. We deduce that

$$\frac{d^2 w}{dF_{ij} dF_{kl}}(F)(Fa)_i b_j (Fa)_k b_l \geq 0$$

for any choice of F, a, b. This is the same as:

$$\frac{d^2 w}{dF_{ij} dF_{kl}}(F) a_i b_j (a_k b_l \geq 0$$

for any F, a, b.

For the group $SL(n, R)$ of matrices with determinant one we obtain a similar condition by imposing the constraint $div\ \eta^s = 0$. This can be done if $a \cdot b = 0$ and $Du(x) \cdot a = 0$. For simplicity suppose that w is defined in a neighbourhood of $SL(n, R)$. Then w is $SL(n, R)$ rank one convex implies

$$(10) \qquad \frac{d^2 w}{dF_{ij} dF_{kl}}(F)(Fa)_i b_j (Fa)_k b_l \geq 0$$

for any $F \in SL(n, R)$, $a, b \in R^n$, $a \cdot b = 0$.

5.1. **Rank one affine functions.** A map w is G rank one affine if w and $-w$ are G rank one convex. For the case $G = GL(n)$ we see that the rank one affines are known. This is very useful in several instances. The reason is that the Euler-Lagrange equation associated to the potential w does not change if one adds a rank affine function to w. At the action functional level

$$I_w(\phi) \; = \; \int_\Omega w(D\phi(x))$$

the addition of a $GL(n, R)$ rank one function means the addition of a closed form which cancels with the integral. This coincidence led to the development of formal calculus of variations in the frame of the jet bundle formalism, which permits to classify all universal conservation laws in elasticity. For this classification see Olver [15].

The case $G = SL(n, R)$ is equally important, because it is about incompressible elasticity. Or, in this case nothing is known, because it is not proven that the $SL(n, R)$ rank one affine functions correspond to closed forms. For this reason Olver's classification [15] of universal conservation laws is not proven to be complete.

We arrived to the following

Open problem: Describe all G rank one affine functions.

In particular situations the problem has been solved. For example if $G = GL(n, R)$ then any rank one affine function is a classical null lagrangian. In the case $SL(2, R)$ we have the following theorem:

Theorem 5.1. *Any $SL(2, R)$ rank one affine function is affine.*

Proof. We have to prove that if $w : SL(2, R) \to R$ is rank one affine then $w(F) = a_{ij}F_{ij} + b$. It is sufficient to prove the thesis for any F in an open dense set in $SL(2, R)$. We shall use the following maps:

$$(X, Y, Z) \in R^* \times R \times R \; \mapsto \; F \; = \; \begin{pmatrix} X & Y \\ Z & \frac{1+YZ}{X} \end{pmatrix}$$

$$(X', Y', Z') \in R^* \times R \times R \; \mapsto \; F \; = \; \begin{pmatrix} \frac{1+Y'Z'}{X'} & Y' \\ Z' & X' \end{pmatrix}$$

Take arbitrary $a = (a_1, a_2)$ and perpendicular $b = (-a_2, a_2)$. If w is SL_2 rank one affine then the mapping

$$t \mapsto f(t; a \otimes b, F) = w(F(1 + ta \otimes b))$$

is linear for any $F \in SL(2, R)$. We have used here the relation and the equality $\exp a \otimes b = 1 + a \otimes b$, for any orthogonal a, b. Rank one convexity of w means that the second derivative of $f(t; a \otimes b, F)$ with respect to t vanishes for any choice of F and a.

We express F in terms of the coordinates $F = F(X, Y, Z)$ and $F = F(X', Y', Z')$. After some elementary computation we obtain the following minimal system of equations for the function $w(X, Y, Z) = w(F(X, Y, Z))$:

$$(11) \qquad \begin{cases} w_{XX}X^2 & = & 2w_{YZ}(1 + YZ) \\ w_{ZZ}X & = & -w_{YZ}Y \\ w_{XY}X & = & -w_{YZ}Z \\ w_{YY} & = & 0 \\ w_{ZZ} & = & 0 \end{cases}$$

From equations (11.4) and (11.5) we find that w has the form

$$w(X, Y, Z) = A(X)YZ + B(X)Y + C(X)Z + D(X)$$

From (11.2) we obtain the equation

$$XC'(X) + XYA'(X) = -A(X)Y$$

From here we derive that $C(X) = c$ and $A(X) = k/X$. We update the form of w, use (11.3) to get $B(X) = b$ and (11.1) to get $D(X) = (k/X) + eX + f$. We collect all the information and we obtain that w has the expression:

$$w(X, Y, Z) = k\frac{1 + YZ}{X} + bY + cZ + eX + f$$

which proves the theorem. $\qquad\square$

Therefore, in the case $G = SL(2, R)$ we have proved that there are no rank one affine functions other than the classical ones. The proof is not adapted to generalizations. The case $G = SL(3, R)$ is open.

Other groups are equally significant, like the group $Sp(n, R)$ of symplectomorphisms. I don't know of any attempt to solve this problem.

5.2. **Rank one convexity and quasiconvexity.** The $GL(n, R)$ rank one convexity is not equivalent to $GL(n, R)$ quasiconvexity in any dimension.

Proposition 5.3. *The function $w : GL(n, R) \to R$ defined by*

$$w(F) \;=\; -\log |\det F|$$

is $GL(n, R)$ rank one convex but not $GL(n, R)$ quasiconvex.

Proof. The map is polyconvex hence it is rank one convex. It is not quasi-convex though. To see this fix $\varepsilon \in (0, 1)$, $A \in GL(n, R)$ and $\Omega = B(0, 1)$. There is a Lipschitz solution to the problem

$$\begin{cases} Dv(x) \in O(n) & \text{a.e. in } \Omega \\ v(x) = \varepsilon x & x \in \partial\Omega \end{cases}$$

We have then, for $u(x) = v(x)/\varepsilon \in [GL(n, R)]^\infty(\Omega)$:

$$\int_\Omega w(ADu(x)) \;=\; \int_\Omega -\log |\det A| \;+\; \int_\Omega n \log \varepsilon \;<\; \int_\Omega w(A)$$

$\qquad\square$

Next proposition justifies this result.

Proposition 5.4. *For any $w : G \to R$ define $\imath w : G \to R$ by:*

$$\imath w(F) = |\det F| \; w(F^{-1})$$

Then w is G rank one convex if and only if $\imath w$ is. Also, if w is G quasi-convex then for any $u \in [G]^\infty(\Omega)$ we have:

$$\int_\Omega w(FDu(x)) \;\geq\; \int_\Omega w(F)$$

Proof. Take u like in the hypothesis. Then for any (continuous) w we have

$$\int_\Omega w(Du^{-1}(x)) \;=\; \int_\Omega \imath w(Du(x))$$

by straightforward computation. Use now the proof of proposition 5.1 to deduce the first part of the conclusion. For the second part use the definition 1.4 and the proposition 3. $\qquad\square$

Let us apply this proposition to $w(F) = -\log |\det F|$. Remark that when $\det F$ goes to zero the function goes to $+\infty$. Now, $\imath w(F) = |\det F| \log |\det F|$ and this function can be continuously prolongated to matrices with determinant zero by setting $\imath w(F) = 0$ if $\det F = 0$. It is easy to see that the prolongation of $\imath w$ ceases to be rank one convex.

6. Application: a class of quasiconvex functions

The goal of this section is to give a class of quasi-convex isotropic functions which seem to be complementary to the polyconvex isotropic ones. We quote the following result of Thompson and Freede [16], Ball [2] (for a proof coherent with this paper see Le Dret [11]).

Theorem 6.1. *Let $g : [0, \infty)^n \to R$ be convex, symmetric and nondecreasing in each variable. Define the function w by*

$$w : gl(n, R) \to R , \quad w(F) = g(\sigma(F)).$$

Then w is convex.

We shall use the Theorem 6.2. Buliga [5]. We need a notation first. Let $x = (x_1, ..., x_n) \in R^n$ be a vector. Then the vector $x^\downarrow = (x_1^\downarrow, ..., x_n^\downarrow) \in R^n$ is obtained by rearranging in decreasing order the components of x. Remark that for any symmetric function $h : R^n \to R$ there exists and it is unique the function $p : R^n \to R$ defined by the relation:

$$p(\sum_{i=1}^{k} x_i^\downarrow) \; = \; h(x_k)$$

Theorem 6.2. *Let $g : (0, \infty)^n \to R$ be a continuous symmetric function and $h : R^n \to R$, $h(x_1, ..., x_n) = g(\exp x_1, ..., \exp x_n)$. Suppose that*

(a) *h is convex,*
(b) *The function p associated to h is nonincreasing in each argument.*

Let $\Omega \subset R^n$ be bounded, with piecewise smooth boundary and $\phi : \overline{\Omega} \to R$ be any Lipschitz function such that $D\phi(x) \in GL(n, R)^+$ a.e. and $\phi(x) = x$ on $\partial\Omega$. Define the function

$$w : GL(n, R)^+ \to R , \quad w(F) = g(\sigma(F))$$

Then for any $F \in GL(n, R)^+$ we have:

$$(12) \qquad \int_\Omega w(FD\phi(x)) \; \geq \; | \, \Omega \, | \, w(F)$$

A consequence of theorem 6.2 and Theorem 2.1 (a) is:

Proposition 6.1. *In the hypothesis of Theorem 6.2, let $\phi_h : \Omega \to R^n$ be a sequence of Lipschitz bounded functions such that*

(a) *for any h $D\phi_h(x) \in GL(n, R)^+$ a.e. in Ω.*
(b) *the sequence ϕ_h converges uniformly to $u : \Omega \to \Omega$, bi-Lipschitz function.*

Then

$$(13) \qquad \liminf_{h \to \infty} \int_\Omega w(D\phi_h(x)) \; \geq \; \int_\Omega w(Du(x))$$

Proof. It is clear that theorem 6.2 implies the hypothesis of point (a), theorem 2.1. Indeed, the conclusion of theorem 6.2 can be written like this: for any $u \in [GL(n, R)^+](\Omega)$ such that

$$\bar{D}u(\Omega) = \frac{1}{| \, \Omega \, |} \int_\Omega Du(x) \; \mathrm{d}x \; \in GL(n, R)^+$$

we have the inequality

$$\int_\Omega w(Du(x)) \; \mathrm{d}x \geq \int_\Omega w(\bar{D}u(\Omega)) \; \mathrm{d}x$$

Take a sequence of mapping $(u_h) \subset [GL(n, R)^+](\Omega)$ uniformly convergent to $F \in GL(n, R)^+$. The previous inequality and the continuity of w imply:

$$\int_\Omega w(F) \ \mathrm{d}x \ \leq \ \int_\Omega w(Du_h(x)) \ \mathrm{d}x$$

Apply now theorem 2.1 (a) and obtain the thesis. $\qquad\qquad\blacksquare$

The class of functions w described in theorem 6.2 and the class of polyconvex functions seem to be different. However, by picking h linear, we obtain a polyconvex function, like

$$w(F) = -\log |\det F|$$

We have seen in proposition 5.3 that this function is not $GL(n, R)$ quasiconvex but proposition 6.1 tells that w is $GL(n, R)^+$ quasiconvex.

We close with an example of another function which we can prove that it is $GL(n, R)^+$ quasiconvex. We use the notation $F = R_F U_F$ for the polar decomposition of $F \in GL(n, R)^+$, with U_F symmetric and positive definite. The example is the function:

$$w : GL(n, R)^+ \to R , \quad w(F) = \det F \ \log(trace \ U_F)$$

With the notation introduced in proposition 5.4, let's look to the the function $\hat{w} = \imath w$. It has the expression:

$$\hat{w} : GL(n, R)^+ \to R , \quad \hat{w}(F) \ = \ \log\left(trace \ U_F^{-1}\right)$$

It is a matter of straightforward computation to check that $\hat{w}$ verifies the hypothesis of theorem 6.2. It is therefore $GL(n, R)^+$ quasiconvex. By proposition 5.4 w is $GL(n, R)^+$ quasiconvex, too, hence lower semicontinuous in the sense of theorem 2.1 (a).

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References

[1] M. Baker, J.L. Ericksen, Inequalities restricting the form of the stress deformation relations for isotropic elastic solids and Reiner-Rivlin fluids, *J. Wash. Acad. Sci.*, **44**, (1954), 33–45

[2] J. M. Ball, Constitutive inequalities and existence theorems in nonlinear elastostatics, in Nonlinear analysis and mechanics: Heriot-Watt Symposium, R. J. Knops (ed.), vol. **1**, *Res. Notes in Math.*, **17**, Pitman, (1977), 187—241

[3] J. M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* , **63**, (1977), 337 — 403

[4] J.M. Ball, J.C. Currie, P.J. Olver, Null Lagrangians, weak continuity and variational problems of arbitrary order, *J. Funct. Anal.*, **41**, (1981), 135 — 174

[5] M. Buliga, Majorisation with applications to the calculus of variations, preprint math.FA/0105044, (2001)

[6] B. Dacorogna, P Marcellini, Implicit partial differential equations, *Progress in nonlinear differential equations and their applications*, **37**, Boston Birkh auser (1999)

[7] D. G. B. Edelen, The null set of the Euler-Lagrange operator, *Arch. Rational Mech. Anal.*, vol. **11**, (1984), 117 — 121

[8] L. Ericksen, Nilpotent energies in liquid crystal theory, *Arch. Rational Mech. Anal.* , vol. **10**, **3**, (1962), 189 — 196

[9] M. Giaquinta, G. Modica, J. Soucek, Cartesian Currents in the Calculus of Variations, *Modern Surveys in Mathematics*, vol. **37-38**, Springer, (1998)

[10] J. Hadamard, Sur une question de calcul des variations, *Bull. Soc. Math. France*, **30**, (1902), 253 — 256

[11] H. Le Dret, Sur les fonctions de matrices convexes et isotropes, *C.R. Acad. Sci. Paris*, **t. 310**, Série I, (1990), 617–620

[12] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorisation and it's Applications, *Mathematics in science and engineering*,**143**, Academic Press, (1979)

[13] N. G. Meyers, Quasi-convexity and lower semicontinuity of multiple variational integrals of any order, *Trans. Amer. Math. Soc.*, vol. **119**, **1**, (1965), 125 — 149

[14] C. B. Morrey, Quasiconvexity and the semicontinuity of multiple integrals, *Pacific J. Math.*, **2**, (1952), 25 — 53

[15] P.J. Olver, Conservation laws in elasticity.I.General results, *Arch. Rational Mech. Anal.*, **85**, (1984), 111–129

[16] R.C. Thompson, L.J. Freede, Eigenvalues of sums of Hermitian matrices III, *J. Research Nat. Bur. Standards B*, **75 B**, (1971), 115–120